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On the regularization for ductile damage models

Kai Langenfeld[1], **Kerstin Möhring**[2], **Frank Walther**[2], and **Jörn Mosler**[1,*]

[1] Institute of Mechanics, TU Dortmund University, Leonhard-Euler-Str. 5, D-44227 Dortmund, Germany
[2] Department of Materials Test Engineering, TU Dortmund University, Baroper Str. 303, D-44227 Dortmund, Germany

Modeling softening behavior often leads to ill-posed boundary value problems. This, in turn, results in pathological mesh-dependent computations as far as the finite-element-method is concerned. A by now widely used technique to eliminate these problems is the micromorphic approach by [1, 2]. This regularization technique implicitly includes gradients of internal variables into the constitutive model. However, and in contrast to other gradient-enhanced models, the micromorphic approach preserves the local structure of the underlying local constitutive model. Within this contribution, it is first shown that a standard application of this approach to ductile damage models does not work. By analyzing the respective equations, a modification of the regularization technique is elaborated – first for an isotropic ductile damage model, i.e. scalar-valued damage. Subsequently, the novel regularization is extended to tensor-valued damage models.

1 Fundamentals

Let $\boldsymbol{u}(\boldsymbol{X})$ be the displacement field of all points $\boldsymbol{X}$ belonging to body $\mathcal{B}$ (the reference configuration). Then, total strain tensor $\boldsymbol{\varepsilon}$ is defined as

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla^T \boldsymbol{u}] \tag{1}$$

where ∇ is the gradient operator with respect to $\boldsymbol{X}$. For modeling plasticity, total strain tensor $\boldsymbol{\varepsilon}$ is additively decomposed into an elastic and a plastic part, i.e., $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$.

Focusing on statics, balance of linear momentum is given in local format by partial differential equation

$$\operatorname{div} \boldsymbol{\sigma} = -\rho_0\, \boldsymbol{B}_0 \quad \forall \boldsymbol{X} \text{ in } \mathcal{B}\,. \tag{2}$$

Here, div denotes the divergence operator, $\boldsymbol{\sigma}$ is the stress tensor and $\rho_0\,\boldsymbol{B}_0$ volume forces. By decomposing the boundary of body $\mathcal{B}$ into disjunct parts $\partial\mathcal{B}_D$ (Dirichlet boundary) and $\partial\mathcal{B}_N$ (Neumann boundary), the boundary conditions associated with partial differential equation (2) can be written as

$$\boldsymbol{u} = \boldsymbol{u}^* \quad \forall \boldsymbol{X} \text{ in } \mathcal{B}_D\,, \qquad \boldsymbol{\sigma}\cdot\boldsymbol{N} = \boldsymbol{T}^* \quad \forall \boldsymbol{X} \text{ at } \partial\mathcal{B}_N\,, \tag{3}$$

in which $\boldsymbol{N}$ is the normal vector at boundary $\partial\mathcal{B}$, and in which $\boldsymbol{T}^*$ are prescribed tractions acting at $\partial\boldsymbol{B}_N$.

2 Local constitutive model – Anisotropic ductile damage

The Helmholtz energy of the anisotropic ductile damage model proposed in [3] is given in simplified manner as

$$\Psi = \frac{\Lambda}{2}\operatorname{tr}(\boldsymbol{b}\cdot\boldsymbol{\varepsilon}^{\mathrm{e}})^2 + \mu\,\boldsymbol{b} : [\boldsymbol{\varepsilon}^{\mathrm{e}}\cdot\boldsymbol{b}\cdot\boldsymbol{\varepsilon}^{\mathrm{e}}]\,, \tag{4}$$

where Λ and μ are the Lamé parameters. The second-order tensor $\boldsymbol{b}$ denotes the integrity of material points, i.e., $\boldsymbol{b} = \boldsymbol{1}$ represents an undamaged state and $\boldsymbol{b} = \boldsymbol{0}$ a fully damaged state.

Following [4], the thermodynamic forces being dual to $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{\mathrm{p}}, \boldsymbol{b})$ follow from Eq. (4) in a straightforward manner as

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = -\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^{\mathrm{p}}} = \Lambda\operatorname{tr}(\boldsymbol{b}\cdot\boldsymbol{\varepsilon}^{\mathrm{e}})\,\boldsymbol{b} + 2\,\mu\,\boldsymbol{b}\cdot\boldsymbol{\varepsilon}^{\mathrm{e}}\cdot\boldsymbol{b}\,, \quad \boldsymbol{\beta} = -\frac{\partial\Psi}{\partial\boldsymbol{b}} = -\Lambda\operatorname{tr}(\boldsymbol{b}\cdot\boldsymbol{\varepsilon}^{\mathrm{e}})\,\boldsymbol{\varepsilon}^{\mathrm{e}} - 2\,\mu\,\boldsymbol{\varepsilon}^{\mathrm{e}}\cdot\boldsymbol{b}\cdot\boldsymbol{\varepsilon}^{\mathrm{e}} \tag{5}$$

where $\boldsymbol{\beta}$ can be interpreted as a tensor-valued energy release rate.

The constitutive model is completed by suitable evolution equations and loading/unloading conditions. Concerning the latter, they are defined by means of a failure/yield surface. Choosing a von Mises type function, yield function

$$\Phi = \sqrt{\sigma_{\mathrm{eq}}} - \sigma_y \quad \text{with} \quad \sigma_{\mathrm{eq}} = \frac{3}{2}\left[\operatorname{tr}([\boldsymbol{b}^{-1}\cdot\boldsymbol{\sigma}]^2) - \frac{1}{3}\operatorname{tr}(\boldsymbol{b}^{-1}\cdot\boldsymbol{\sigma})^2\right] \tag{6}$$

* Corresponding author: e-mail joern.mosler@tu-dortmund.de, phone +49 231 755 5744, fax +49 231 755 2688

is adopted, where σ_{eq} is a generalized equivalent von Mises stress measure. Finally, the evolution equations of the model are defined. For that purpose, the framework of Generalized Standard Materials, cf. [5], is applied. Within this framework, evolution equations are defined as gradients of a potential g. To be more precise,

$$\dot{\varepsilon}^{\mathrm{p}} = \lambda\,\partial_{\boldsymbol{\sigma}} g\ , \quad \dot{\boldsymbol{b}} = \lambda\,\partial_{\boldsymbol{\beta}} g\ , \tag{7}$$

where $\lambda \geq 0$ is the plastic multiplier. In what follows, potential g is chosen as

$$g = \Phi(\boldsymbol{\sigma};\, \boldsymbol{b}) + \Gamma_{\boldsymbol{\beta}}(\boldsymbol{\beta};\, \boldsymbol{b}) \quad \text{with} \quad \Gamma_{\boldsymbol{\beta}}(\boldsymbol{\beta};\, \boldsymbol{b}) = \frac{\eta_1}{2}\,\mathrm{tr}(\boldsymbol{b}\cdot\boldsymbol{\beta})^2 + \frac{\eta_2}{2}\,\boldsymbol{b} : [\boldsymbol{\beta}\cdot\boldsymbol{b}\cdot\boldsymbol{\beta}]\ . \tag{8}$$

Accordingly, the flow rule is associative, i.e., $\dot{\varepsilon}^{\mathrm{p}} = \lambda\,\partial_{\boldsymbol{\sigma}}\Phi$. Furthermore potential (8) which governs the damage evolution, depends on the dual variables $\boldsymbol{b}$ and $\boldsymbol{\beta}$. Here, model parameter η_1 defines the isotropic part of the damage evolution and η_2, in turn, the anisotropic part.

3 Micromorphic regularization of the constitutive model

In this section, the local anisotropic damage model presented before is regularized by means of a gradient formulation.

3.1 Standard approach

The softening behavior of the anisotropic damage model introduced in Section 2 is determined by the monotonic decreasing second-order tensor $\boldsymbol{b}$ (its eigenvalues decrease and converge towards zero). In order to regularize the local constitutive model, the gradient of $\boldsymbol{b}$ shall be incorporated into the model. Following [1, 2], the regularization is not directly applied to $\boldsymbol{b}$, but to an additional second-order tensor $\boldsymbol{\varphi_b}$. With this tensor, a micromorphic approximation of a standard gradient regularization is obtained by means of enhanced potential

$$\Pi^{\mathrm{enh}} = \int\limits_{\mathcal{B}} \Psi^{\mathrm{enh}}\,\mathrm{d}V = \int\limits_{\mathcal{B}} \left[\Psi + \Psi^{\mathrm{pen}} + \Psi^{\mathrm{grad}}\right]\,\mathrm{d}V\ , \quad \Psi^{\mathrm{pen}} = \frac{c_{\boldsymbol{b}}}{2}\|\boldsymbol{\varphi_b} - \boldsymbol{b}\|^2\ , \quad \Psi^{\mathrm{grad}} = \frac{l_{\boldsymbol{b}}}{2}\|\nabla\boldsymbol{\varphi_b}\|^2\ . \tag{9}$$

Here, Ψ denotes the Helmholtz energy of the unregularized underlying local damage model presented in the previous section and $l_{\boldsymbol{b}}$ is a length parameter which defines the zone showing localized material damage. If energy minimization governs the evolution of field $\boldsymbol{\varphi_b}$ and if, furthermore, penalty parameter $c_{\boldsymbol{b}}$ is chosen as sufficiently large, $\boldsymbol{\varphi_b}$ converges towards $\boldsymbol{b}$ and both spatial distributions coincide.

Assuming energy minimization once again as the overriding principle which governs the evolution of $\boldsymbol{\varphi_b}$, the gradient regularization results from stationary of potential (9). To be more explicit,

$$\delta_{\boldsymbol{\varphi_b}}\Pi^{\mathrm{enh}} = \int\limits_{\mathcal{B}} c_{\boldsymbol{b}}[\boldsymbol{\varphi_b} - \boldsymbol{b}] : \delta\boldsymbol{\varphi_b} + l_{\boldsymbol{b}}\,\nabla\boldsymbol{\varphi_b} \,\vdots\, \nabla\delta\boldsymbol{\varphi_b}\,\mathrm{d}V\ , \tag{10}$$

which can be transformed into Laplace type

$$l_{\boldsymbol{b}}\,\mathrm{div}\,\nabla\boldsymbol{\varphi_b} = c_{\boldsymbol{b}}\,[\boldsymbol{\varphi_b} - \boldsymbol{b}]\,. \tag{11}$$

Its insertion into $\boldsymbol{\beta}^{\mathrm{enh}}$ leads to

$$\boldsymbol{\beta}^{\mathrm{enh}} = -\frac{\partial\Psi^{\mathrm{enh}}}{\partial\boldsymbol{b}} = \underbrace{-\frac{\partial\Psi}{\partial\boldsymbol{b}}}_{=\boldsymbol{\beta}} - \frac{\partial\Psi^{\mathrm{pen}}}{\partial\boldsymbol{b}} = \underbrace{\boldsymbol{\beta}}_{\text{local model}} + l_{\boldsymbol{b}}\,\mathrm{div}\,\nabla\boldsymbol{\varphi_b} \tag{12}$$

and, hence, the gradient extension becomes obvious.

Although the model is indeed gradient-enhanced, it does — surprisingly — not regularize the local damage model.

3.2 Novel Approach

According to the results presented in Section 5 for a one-dimensional truss with an imperfection, the naive micromorphic enhancement only affects the imperfect element. In order to implement an influence of $\boldsymbol{\varphi_b}$ on yield function Φ for the neighboring elements as well — and thus on their damage evolution — variable

$$\boldsymbol{\omega_b} = \frac{\partial\Psi^{\mathrm{enh}}}{\partial\boldsymbol{\varphi_b}} = c_{\boldsymbol{b}}\,[\boldsymbol{\varphi_b} - \boldsymbol{b}] \overset{\text{Eq. (11)}}{=} l_{\boldsymbol{b}}\,\mathrm{div}\,\nabla\boldsymbol{\varphi_b} \tag{13}$$

being dual to $\boldsymbol{\varphi_b}$ is introduced. Evidently, this variable includes a gradient contribution. Furthermore, $\boldsymbol{\omega_b}$ connects the local energy release rate tensor $\boldsymbol{\beta}$ and its gradient-enhanced counterpart $\boldsymbol{\beta}^{\text{enh}}$ according to

$$\boldsymbol{\beta}^{\text{enh}} = \boldsymbol{\beta} + \boldsymbol{\omega_b} \; . \tag{14}$$

With this notation, the yield function of the previous micromorphic model is replaced/enhanced by

$$\Phi^{\text{enh}} = \Phi + f_{\boldsymbol{\omega_b}} \qquad \text{with} \qquad f_{\boldsymbol{\omega_b}}(\boldsymbol{\omega_b}) = -\sqrt{\boldsymbol{b}^{-1}} : \boldsymbol{\omega_b} \; . \tag{15}$$

It bears emphasis that the enhancement does not need to be additively in structure. Choice (15) can be interpreted as a gradient dependent yield limit, i.e., $\sigma_y^{\text{enh}} = \sigma_y - f_{\boldsymbol{\omega}}$.

Clearly, the yield function of the original local model and that of the modified counterpart according to Eq. (15) are different — as long as $f_{\boldsymbol{\omega_b}} \neq 0$, i.e., $\boldsymbol{b} \neq \boldsymbol{\varphi_b}$. In order to keep the damage potential of the original local model, damage potential $\Gamma_{\boldsymbol{\beta}}$ is replaced by (compare to Eq. (8))

$$\begin{aligned} \Gamma_{\boldsymbol{\beta}} &= \frac{\eta_1}{2}\,\text{tr}(\boldsymbol{b}\cdot[\boldsymbol{\beta}^{\text{enh}} - \boldsymbol{\omega_b}])^2 + \frac{\eta_2}{2}\,\boldsymbol{b} : [[\boldsymbol{\beta}^{\text{enh}} - \boldsymbol{\omega_b}]\cdot\boldsymbol{b}\cdot[\boldsymbol{\beta}^{\text{enh}} - \boldsymbol{\omega_b}]] \\ &= \frac{\eta_1}{2}\,\text{tr}(\boldsymbol{b}\cdot\boldsymbol{\beta})^2 + \frac{\eta_2}{2}\,\boldsymbol{b} : [\boldsymbol{\beta}\cdot\boldsymbol{b}\cdot\boldsymbol{\beta}] \, . \end{aligned} \tag{16}$$

4 Prototype model — Regularized isotropic ductile damage

The damage model elaborated in the previous sections can be applied to a broad range of materials. However, if large engineering structures are to be numerically analyzed, isotropic damage models are sometimes sufficient. Clearly, isotropic damage models are — from a numerical point of view — significantly more efficient. For instance, as far as the micromorphic gradient regularization is concerned, the anisotropic model requires six additional degrees of freedom (the second-order integrity tensor is symmetric, i.e., $\boldsymbol{b} = \boldsymbol{b}^T$), whereas an isotropic model would require only one additional degree of freedom. Due to the aforementioned reasons, an isotropic approximation of the anisotropic damage model presented before is given here.

In line with isotropic material damage, the integrity tensor is postulated to be of type

$$\boldsymbol{b} = b\,\boldsymbol{I} \; . \tag{17}$$

With assumption (17) the resulting Helmholtz energy of the isotropic damage model reads

$$\Psi = \frac{\Lambda}{2}\,b^2\,\text{tr}(\boldsymbol{\varepsilon}^{\text{e}})^2 + \mu\,b^2\,\boldsymbol{\varepsilon}^{\text{e}} : \boldsymbol{\varepsilon}^{\text{e}} \; . \tag{18}$$

Analogously to potential (9), the micromorphic enhanced potential for an isotropic material degradation follows as

$$\Pi^{\text{enh}} = \int\limits_{\mathcal{B}} \Psi^{\text{enh}}\,\text{d}V = \int\limits_{\mathcal{B}} \left[\Psi + \Psi^{\text{pen}} + \Psi^{\text{grad}}\right]\,\text{d}V \; , \quad \Psi^{\text{pen}} = \frac{c_b}{2}\,[\varphi_b - b]^2 \; , \quad \Psi^{\text{grad}} = \frac{l_b}{2}\|\nabla\varphi_b\|^2 \tag{19}$$

and hence, the thermodynamic forces take the form

$$\boldsymbol{\sigma} = \frac{\partial\Psi^{\text{enh}}}{\partial\boldsymbol{\varepsilon}} = \Lambda\,b^2\,\text{tr}(\boldsymbol{\varepsilon}^{\text{e}})\,\boldsymbol{I} + 2\,\mu\,b^2\,\boldsymbol{\varepsilon}^{\text{e}} \; , \quad \omega_b = \frac{\partial\Psi^{\text{enh}}}{\partial\varphi_b} = c_b\,[\varphi_b - b] = l_b\,\text{div}\,\nabla\varphi_b \; , \tag{20}$$

$$\beta^{\text{enh}} = -\frac{\partial\Psi^{\text{enh}}}{\partial b} = -\Lambda\,b\,\text{tr}(\boldsymbol{\varepsilon}^{\text{e}})^2 - 2\,b\,\mu\,\boldsymbol{\varepsilon}^{\text{e}} : \boldsymbol{\varepsilon}^{\text{e}} + c_b\,[\varphi_b - b] = \beta + \omega_b \; . \tag{21}$$

Likewise, assumption (17) and micromorphic modifications (15) and (16) affect the plastic potential. To be more explicit,

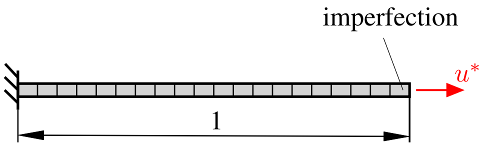
$$\Phi^{\text{enh}} = \sqrt{\sigma_{\text{eq}}} - \tau_y + f_{\omega_b} \quad \text{with} \quad \sigma_{\text{eq}} = \frac{3}{2}\,b^{-2}\,\boldsymbol{\sigma}_{\text{dev}} : \boldsymbol{\sigma}_{\text{dev}} \, , \quad f_{\omega_b} = -\sqrt{b^{-1}}\,\omega \tag{22}$$

$$\Gamma_\beta = \frac{\eta}{2}\,b^2\,[\beta^{\text{enh}} - \omega_b]^2 = \frac{\eta}{2}\,b^2\,\beta^2 \; , \tag{23}$$

where the modification can be interpreted as a gradient-dependent yield stress.

5 Numerical Example — Imperfect element test (1D)

A one-dimensional truss is investigated in this subsection. The truss, which is fixed on the left hand side, is loaded in tension by prescribing the displacement at the right hand side. The mechanical setup, the finite element discretization by means of



Material parameter				
E	σ_y	η	c_b	l_b
400	1 (0.95)	2000	5	100

Fig. 1: One-dimensional problem — truss with imperfection: Description of the boundary value problem, finite element discretization by means of (20) finite elements and model parameters (the discretizations by means of 40, 80 and 160 finite elements are not shown here)

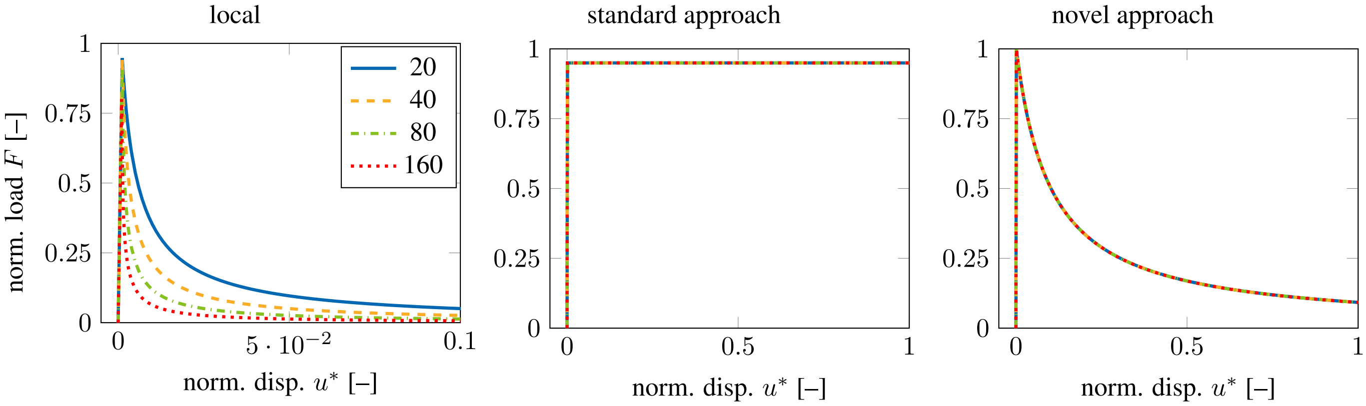


Fig. 2: One-dimensional problem — truss with imperfection: Load-displacement diagrams predicted by: (left) the local model (the different scale has been chosen in order to be able to distinguish the different curves), (middle) the standard micromporphic model and (right) novel micromorpic model for different finite element meshes (20, 40, 80 and 160 finite elements)

(20, 40, 80 and 160) linear displacement-driven finite elements, together with the model parameters, are summarized in Fig. 1. In order to trigger localization, the element on the right hand side is weakened by reducing its initial yield stress by 5%.

The mechanical responses as predicted by the three different models are summarized in Fig. 2. Accordingly, the underlying local model as well as its modified micromorphic extension capture the desired softening response. By way of contrast, the standard straightforward micromorphic model does not, but shows a plateau in the force-displacement diagram. It bears emphasis that the spatial distribution of b is not constant for the standard straightforward micromorphic model, i.e., localized damage indeed occurs within the element showing the imperfection. However, since fields b and φ_b are only one-directionally coupled, b does not evolve in the neighboring elements. Furthermore, the micromorphic extension eliminates the evolution of b within the localized element such that the total mechanical response is not characterized by material softening anymore.

For the sake of completeness, mesh objectivity of the novel model is shown in Fig. 2 (right). While the load-displacement diagrams associated with the local model depend on the underlying finite element mesh, this pathological behavior is neither observed for the standard micromorphic nor for the novel micromorphic model. However and as already mentioned before, only the novel micromorphic model captures the desired softening response.

It bears emphasis that the observed behavior is restricted to the one-dimensional case. To be more precise, the elaborated damage model combined with the standard approach of the micromorphic regularization does capture softening behavior in 2D as well as in 3D, but the model remains ill-posed from a mathematical point of view, i.e., the pathological mesh dependence can be seen in 2D and in 3D — in sharp contrast to the novel model.

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